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AUTHOR Davis, Robert B.  
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ABSTRACT

This paper, presented at the 1979 meeting of the American Educational Research Association (AERA), investigates student errors in high school mathematics. A conceptual framework of hypothetical information-handling processes such as procedures, frames, retrieval from memory, visually-mediated sequences (VMS sequences), the integrated sequence, and meta-language are suggested to interpret the observations of students learning arithmetic, algebra, geometry, and calculus. Some of the errors discussed are: (1) random errors vs. regular errors; (2) VMS sequence errors; (3) frame errors; (4) language-related errors; (5) abstract verbal statements; (6) epistemological errors; and (7) the persistence of incorrect rules of procedures. (HM)

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ERROR ANALYSIS IN HIGH SCHOOL MATHEMATICS,  
CONCEIVED AS INFORMATION-PROCESSING PATHOLOGY<sup>1</sup>

Robert B. Davis  
University of Illinois, Urbana/Champaign

As the (slightly revised) title suggests, we conceive the learning of high school mathematics as a matter of information processing, and for more than five years we<sup>2</sup> have been studying student errors as pathology in this information processing.

1. A Few Key Concepts. Before considering errors, we list a few of the key hypothetical components in our conceptualization of information processing, especially the VMS sequence, integrated sequence, frame (or schema), and meta-language.

- a. The Procedure. A procedure is a definite sequence of steps, aimed at achieving some quite specific result. (Think of it as a human information-processing analogue of a program in a programmable hand-held calculator.) A procedure can relate to other procedures in several ways, the most important being that Procedure A can

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<sup>2</sup>In addition to the author, the main investigators have been Curtis McKnight, Steve Young, Elizabeth Jockusch, Myriam Steinback, Stanley Erlwanger, and Katie Reynolds Hannibal.

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call upon Procedure B, in order to accomplish part of the assignment of Procedure A. In such a case, Procedure B is said to be a sub-procedure of Procedure A, and A is said to be a super-procedure. A well-known result from Erlwanger's studies of 4th, 5th, and 6th grade arithmetic students is that their arithmetic errors are nearly always located in super-procedures; the various sub-procedures, in the main, tend to be reasonably reliable.

- b. The VMS Sequence. Many mathematical tasks have a certain quasi-periodic structure: one sees something, which leads to the retrieval and execution of some procedure; the execution of this procedure yields a modified visual input, which leads to the retrieval and execution of the next segment of procedures, and so on. One thus has a sequence of visual input, procedure execution, new visual input, procedure execution,....Such a pattern is more or less clearly recognizable in, say, a long division problem in arithmetic, or a "long" multiplication problem, or in the usual written algorithms for multiplying or dividing polynomials. [It is also familiar outside of mathematics, e.g., in driving somewhere if the route is complicated and is relatively unfamiliar. One hopes to recognize a landmark - oh, yes, at that school building one turns right - then to drive along until another key landmark triggers the next key decision and so on. One could not describe the entire route merely from memory, without the prompts given by key visual inputs along the route.] Sequences of this type

have come to be called visually-moderated sequences ("VMS" sequences).

- c. The Integrated Sequence. Successful mathematics students, through practice, are able to develop many VMS sequences into holistic entities which they can contemplate without going through the entire sequence, step by step. (Similarly, our driver becomes able, with practice, to describe the entire route in detail - and hence can instruct others in exactly how to follow the route - without the need to physically re-trace the route in order to prompt his memory.)

Sequences which have reached this state are called integrated sequences. [For most present readers, long division is probably an example of an integrated sequence. One can "see the whole process in the mind's eye" - at least in general - without actually writing it down on paper. The specific numerals probably must be suppressed, but the general pattern can be visualized clearly.]

- d. Frames, Scripts, or Schemata. Although differences are distinguishable between the precise usages of Minsky, Schank, Rumelhart and Ortony, etc., no distinction will be made here, either in name ("frame," "script," or "schema") or in detail description. What is common to all three approaches is far more important than their differences.

Several considerations prompted the introduction of frames. For example, consider the behavior related to "understanding" a written passage, such as:

It was Paul's birthday. Jane and Alex went to buy presents. "Oh look," Jane said, "I'll get him a kite." "He already has one," Alex said. "He'll make you take it back."

Most U.S. adults, having read that passage, can answer questions such as:

- Q1: Why are Jane and Alex buying presents?
- A1: Because it's Paul's birthday.
- Q2: For whom are the presents intended?
- A2: Paul.
- Q3: Where did Jane and Alex go?
- A3: Presumably to a store that sells kites and other things children like.
- Q4: The next to the last word in the paragraph - "it" - to what does that refer?
- A4: To the kite that Jane was proposing to buy.

Having answered such questions, most humans are surprised to find that not one of those answers is given explicitly in the written passage. The answers have come from the reader's general knowledge of birthdays, presents, stores, etc. In Minsky's language, the reader "retrieved a birthday frame." Frames play an important role, because they embody a representation of a small part of the structure of the world. They are hypothesized to involve variables, with default procedures for assigning values to variables when input data fails to do so. [Asked to repeat the story a week or two later, most adults will unwittingly add to it, by inserting information from the "birthday frames" that was not included in the original story.]

There is a growing body of evidence suggesting the operation of some mechanism generally similar to "frames." The response to natural-language text is among the strongest. In addition to text studies like "birthday," consider also the well-known phenomenon of person A claiming that B told

him fact X, when B denies having done so.

(For data specific to mathematics, cf. the study by Hinsley, et.al., which we discuss below.)

- e. Meta-Language. Confronted by a problem, a student may need to retrieve one or more frames, and sequence them properly. Possession of appropriate descriptors (a "meta-language") for these frames can help. Consider the problem:

Find the distance from the origin  $O(0,0,0)$  to the plane through the points  $A(5,0,0)$ ,  $B(0,3,3)$  and  $C(0,0,2)$ .

This can be a hard problem, but if one has both the necessary schemata for various parts of the problem, and appropriate descriptors to help get them selected and interrelated. If, for example, one knows that the dot product  $\vec{u} \cdot \vec{V}$  between any vector  $\vec{V}$ , and a unit vector  $\vec{u}$  in some special direction, gives us the component of  $\vec{V}$  in the direction of  $\vec{u}$ , one can find any vector from  $O$  to the plane (for example, the vector  $(5,0,0)$ ), and dot it into a unit normal vector. If one has any non-zero normal vector  $\vec{N}$ , one can get a unit normal vector by dividing  $\vec{N}$  by its length. If one has the equation of the plane in the form  $ax+by+cz=d$ , one easily gets a normal vector  $\vec{N}$ . And so on. Given descriptors such as these, the problem of selecting and sequencing the necessary schemata becomes entirely manageable.

- f. Frame Adequacy Judgements. When a frame is selected and retrieved, and when a mapping of input data into frame variables has been attempted, there must be a mechanism for deciding either that the "fit" is a satisfactory one, or else that it is not.

2. Random Errors vs. Regular Errors. Are errors random, or do they exhibit patterns? Work of Erlwanger and others, and more recently work by John Seely Brown, Marilyn Matz, and their colleagues, have indicated that a high percentage of errors are in fact systematic. In any case, although genuinely "random" errors presumably exist, the errors that are most important, both for students and for teachers, are the regular ones. Students recognize this fact intuitively; one of the most common student requests is: "Tell me what I'm doing wrong." They thus conceptualize the difficulty, as Brown and others do, in terms of one or more procedures that have one or more "bugs" in them. But "buggy" procedures imply systematic errors. Of course we should all take the care that reduces random errors; but de-bugging flawed procedures is one of the main tasks in learning.

3. VMS Sequence Errors. The world at large - and a very sizeable per cent of students - imagine all mathematics to resemble long division: a "cook book" recipe that you learn and follow by rote - that is to say, a VMS sequence. Mathematics, for the man in the street, is a collection of VMS sequences.

Erlwanger has found that the vast majority of VMS sequence errors made by the 4th, 5th, and 6th grade arithmetic students whom he has studied were made in super-procedures that were the subject of recent learning, and not in the sub-procedures which were called for by these super-procedures. [Erlwanger, 1975; Davis, 1977.]

A typical example:  $.3 + .4 = .07$

The sub-procedure that inputs 3 and 4, and outputs 7, performed correctly; the sub-procedure that counted decimal places and added them worked correctly; but, of course, this last sub-procedure ought not to have been activated for this problem.

Particularly interesting are the uni-directional "binary switches" [Davis, Jockusch, and McKnight, 1978], such as  $4 \times 4 = 8$ .

Errors of this type have the following attributes:

- i) Procedure B is used when Procedure A should have been used [in the example, addition was used when multiplication should have been];
- ii) The notations for Procedure A and for Procedure B are visually similar [ + vs.  $\times$  ];
- iii) The procedure used was learned earlier in the student's life than the procedure that was slighted - an earlier procedure erroneously displaced a "newer" one [addition is learned before multiplication in nearly all school curricula];
- iv) The switch (presumably because of (iii) ) is one-directional: The older replaces the newer, never conversely [we have taped no instances of  $4 + 4 = 16$ , or  $2 \times 3 = 8$  (but one often finds  $2^3 = 6$ )].

These uni-directional errors have an interesting theoretical explanation, in terms of the discriminations that are (or are not) made internally within procedures and super-procedures [cf. Davis et. al., op. cit.]. Briefly, it is postulated that the discriminations which are made are no better than what is required at that time; hence, future situations arise (given new concepts, definitions, procedures, or notations) for which adequate discriminations have not been made a part of internal labelling processes.

Among the most common of all errors are distributive law errors, reported by many observers. An example:

$$c(x^2 + y^2) = cx^2 + y^2.$$

Similar to distributive law errors are grouping errors, as in the following example, from a 10th grader's attempt to obtain the equation of an ellipse from the two-foci definition:

$$c \sqrt{(x-a)^2 + y^2}$$

was squared to get, erroneously,

$$c^2 (x-a)^2 + y^2.$$

The trouble here, clearly, is that the square root sign played two roles, only one of which has been recognized by this student: it indicates that a square root is to be taken, but it also indicates grouping in the sense of parentheses.

Many other specific errors in VMS sequences can be identified - for example, the error common among beginning algebra students of combining like terms as in

$$2x^3 + 5x^2 + x^3 + 2x^2 + 3x^3$$

to get

$$7x^2 + 5x^3,$$

treating the "invisible" coefficient one as if it were zero. In fact, this is a predictable error, the consequence of using a procedure like this:

```

10  set n
20  identify next term xn
30  read off numeral to the left of xn
40  add into sum for coefficient of xn
50  loop back to line 20 as needed
60  reset n
70  go to line 20, unless done.
```

4. Frame Errors. One can easily find instances of frame errors of several types, but they can all be sorted into two broad categories: failure to retrieve a frame, vs. the retrieval of an inappropriate frame.

Central to questions of frame retrieval is the "top down" vs. "bottom up" distinction. When a frame has been retrieved and judged appropriate (at least pro tempore), information processing can be directed by the structure present in the frame. Having retrieved a frame for quadratic equations, we know to look for the coefficients, to see whether the discriminant is positive, negative, zero, etc., because the frame indicates the needed information. Having retrieved a frame for simultaneous linear equations, we know to ask whether the determinant of the coefficients is zero or not, because the frame asks for this data. Processes shaped by the demands of a frame in such a way are often called "top-down." The frame directs our processing, we "know where to look and what to look for."

By contrast, if we have some information processing task prior to frame retrieval, some bootstrapping operation must get us from "primitive" data to the stage of retrieving an appropriate frame. This is often called "bottom up" processing.

Presumably at the beginning of an information processing task - to the extent that a "beginning" can exist - processing must at first be "bottom up." One would not know, at first, which frame to retrieve (of course this can be modified by mental set, etc.). The transition to "top down" processing can occur very quickly, as Hinsley, et. al., demonstrated dramatically. In one experiment, for example, it was found that, for some subjects, merely the noun phrase "A river steamer..." was sufficient to trigger the retrieval of an appropriate schema [indicated by the subject

saying "It's going to be one of those river things with upstream, downstream, and still water. You are going to compare times upstream and downstream - or if the time is constant, it will be the distance."].

a. Retrieval Errors - Failure to Retrieve

Failure to retrieve any candidate frame whatsoever can occur if the individual has no appropriate frame stored in memory - surely one of the most common of all failures. It explains why most of us don't know most of the things we don't know.

But even if there is an appropriate frame stored in memory, one can still fail to retrieve it. B, a 9th grade student, was trying to solve a system of simultaneous equations in  $\cos\theta$  and  $\sin\theta$ .

After a protracted period of failing to solve the problem, B gave an exclamation, did solve the problem, and said "when I stopped thinking about trig and started thinking about systems of equations" it became easy. Moral: one's memory is somehow partitioned. Searching the wrong sections will usually be unproductive.

b. Retrieval Errors - Retrieving the Wrong Frame. But

retrieving an inappropriate frame can be a particularly virulent form of bad news. P, a 10th grade student meeting differentiation for the first time, spent two weeks dealing with information processing tasks such as learning the formula

$$\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx} \quad (1)$$

and solving the problem of finding  $\frac{dy}{dx}$  if  $y = x^4 + 3x^2 + 8x$ , by retrieving frames dealing with variables, fractions, and the distributive law which were entirely unsuitable.

For (1), he assumed "d" and "x" were variables just like those he had dealt with before. Hence he assumed that (1) was nothing more than the distributive law. For the second problem, given

$$y = x^4 + 3x^2 + 8x$$

he divided by x to get

$$\frac{y}{x} = x^3 + 3x + 8,$$

then multiplied  $\frac{y}{x}$  by 1 in the form of  $\frac{d}{d}$ ,

to get the (wrong) result

$$\frac{dy}{dx} = x^3 + 3x + 8.$$

In the sequence

$$\frac{dy}{dx} = \frac{d}{dx} (x^3 + 5x + 2) \quad (i)$$

$$= \frac{d}{dx} (x^3) + \frac{d}{dx} (5x) + \frac{d}{dx} 2 \quad (ii)$$

$$= 3x^2 + 5, \quad (iii)$$

whence  $y' = 3x^2 + 5, \quad (iv)$

P objected that the  $\frac{d}{dx}$  had gotten lost, and should appear in line (iv).

When circumstances finally compelled him to recognize that his whole approach was quite wrong, he was, unsurprisingly, devastated. The retrieval of an inappropriate frame - and its use for two weeks - had left him far behind his classmates. There must have been a growing accumulation of doubt-casting

pieces of information, but for two weeks he had continued, presumably by heroic efforts, to try to fit everything into his purely algebraic frames.

One can get an empathetic comprehension of P's plight by imagining a student confronted by the Cayley-Hamilton theorem, which states that the matrix  $A$  satisfies its own characteristic equation,

$$\det (A - \lambda I) = 0. \quad (v)$$

If one replaces  $\lambda$  by  $A$  in equation (v), it seems that the Cayley-Hamilton theorem is a complete triviality:  $AI=A$ , and  $A - A$  is the zero matrix, and the determinant of the zero matrix is surely 0. What's all the fuss about?

Nothing blocks the search for an appropriate frame more effectively than the retrieval of a frame that (perhaps incorrectly) passes the adequacy test. Those who are satisfied do not seek.

- c. Retrieval Errors - Semantic Templates. R, an adult studying college algebra at a community college, was confronted by the need to compute

$$(\sqrt{x+a} - \sqrt{x-a})^2$$

by carrying out the squaring. When the instructor referred to  $(A+B)^2$ , R considered this an irrelevant digression, and did not see how to make use of it. This was a typical response by R. Apparently R has not recognized that one has certain key patterns in one's mind - call them semantic templates - and that mathematics sometimes requires an effort to match up some notation on paper with an appropriate semantic template.

R feels that something explicitly present in

$$(\sqrt{x+a} - \sqrt{x-a})^2$$

should determine the next step. In effect, she expects all of mathematics to resemble the long division algorithm, to be a rote VMS sequence.

A slightly similar, though far less extreme, situation is readily observable among tenth grade students trying to find the equation of an ellipse from the "sum of the distances from two fixed points" definition; one easily gets, by a step-by-step process, to the result

$$a^2 x^2 + c^4 = c^2 x^2 + a^2 c^2 + c^2 y^2. \quad (\text{vi})$$

Nothing in this line clearly determines what should be done next. To get further, one must shift gears and begin to think of the desired result. If one knows that the goal is an equation of the form

$$\frac{x^2}{R^2} + \frac{y^2}{W^2} = 1, \quad (\text{vii})$$

for some R and some W, then getting from equation (vi) to the final result is easy - but what is required is a change from step-to-step processing, switching instead to a strategy of keeping the goal clearly in mind.

- d. When You Do NOT Want Frames. To some extent one ordinarily considers a frame-like organization of knowledge to be more mature, and more secure, than sequential organization. Of course this view overlooks the cyclic reality of small pieces being "chunked" together [Miller], thus getting a new collection of primitive elements, soon to be in need of further chunking.

It also ignores another reality: sometimes rigid

sequential processing is to be preferred as more appropriate than frame-like or "gestalt" processing.

E, a 10th grade geometry student of high ability, for whom English was not a native language, had studied the focus-directrix definition of the parabola, and the two-focus definition of the ellipse and hyperbola. On an examination the class faced a novel task: finding the equation of the paths of a point that moves so that its distance from a fixed point is always half its distance from a fixed line. The rest of the class proceeded as expected: writing the equation of distances, using the distance formula, squaring, and simplifying. At this point the best students were surprised; they had expected a parabola (from the "point-line" form), but recognized that they had obtained the equation of an ellipse. E proceeded entirely differently. She identified specific points on the path, recognized that the path was apparently an ellipse, wrote (from memory)

$$\frac{x^2}{c^2} + \frac{y^2}{b^2} = 1,$$

and found c and b by considering recognizable points on the ellipse.

What had E done? In effect, instead of using a step-by-step procedure, she retrieved a frame, made appropriate replacements for frame variables, and proclaimed the result.

This was not what had been intended, and was in fact not a proof, yet it was not entirely without merit: guessing the form of an answer, and subsequently determining the details

(such as specific values for variables) may well be the most common general method in all of applied mathematics.

#### 5. Language-Related Errors.

Certain errors are seemingly related to ambiguities in, or poor use of, natural language.

Jennifer, a third grade girl, claimed that the written expression

$$804$$

and the written expression

$$84$$

were the same, because "adding zero makes no difference; zero means nothing."

She similarly argued for no difference between

$$8$$

and

$$80$$

and

$$800,$$

when all three were presented to her simultaneously in written form.

Similarly, a common error among beginning algebra students is to pronounce

$$2x^2 - x^2$$

as "two xsquared, take away x squared." Students who read it this way nearly always write the result as

$$2x^2 - x^2 = 2.$$

#### 6. Abstract Verbal Statements.

Think for a moment of a typical computer program (NOT a sophisticated text-interpreting A.I. program). If you type in "to integrate

$$\int \frac{dx}{x^2 + \sqrt{x^2 + a^2}}, \text{ if } a > 0,$$

rationalize the denominator," and if you then type in the problem

$$\int \frac{dx}{x^2 + \sqrt{x^2 + a^2}},$$

nothing useful will result. In some sense the computer "has the necessary information in its memory," but the computer program has no method for using the abstract verbal formulation "to integrate

$$\int \frac{dx}{x^2 + \sqrt{x^2 + a^2}}, \quad \text{if } a > 0,$$

rationalize the denominator" as the basis for constructing a computer program capable of executing this procedure.

Thus the specific problem of Jennifer, or of " $2x^2$  take away  $x^2$ ", is really the tip of a much more interesting iceberg: to get from an abstract verbal formulation to an executable step-by-step procedure is no small undertaking. How can humans ever accomplish it?

We conjecture that they do not. We conjecture that an abstract verbal formulation ("AVF") can serve as a guide to the creation of an executable program ONLY if the AVF is a meta-language statement of the type discussed earlier. In particular this means that the student must (i) have had so much experience with the relevant procedures, and (ii) must have thought about them so much, that the procedures have acquired reliable descriptors in the sense of an appropriate meta-language.

7. Epistemological Errors. Davis, Jockusch, and McKnight list a collection of "epistemological" errors. We cite one type. Students notoriously have concerns about the propriety of defining  $0!$  as 1, or of defining  $A^{\frac{1}{2}}$  as  $\sqrt{A}$ , or of defining  $A^{-1}$  as  $\frac{1}{A}$ . This difficulty comes mainly from a failure to recognize that they are building up a system of mathematics. At any given point the system is only partially

completed. There are "frontiers" where further definitions have not yet been made (although appropriateness criteria for future constructions may be implied by what has already been built).

This is an especial problem for younger students, presumably because they have not yet lived long enough to realize that their city, their state, their nation, their laws, their traditions, their literature, their culture - and, indeed, themselves - are still being built. None is yet finished.

#### 8. The Persistence of Incorrect Rules of Procedure.

Students find it convenient to simplify

$$(\sqrt{(x+a)^2 + y^2} + \sqrt{(x-a)^2 + y^2})^2$$

as  $(x+a)^2 + y^2 + (x-a)^2 + y^2,$

a procedure which has elegance, economy, and only the single flaw of being incorrect.

Incorrect rules of this type have a nearly unbelievable persistence - or else they are repeatedly regenerated by some process. They pose one of the main obstacles to progress in algebra. It is worth inquiring into the kind of information processing that may be involved in their dandelion-like tenacity, and in attempts at rooting them out.

To begin with, is the "linear squaring simplification"

$$(\sqrt{A} + \sqrt{B})^2 = A + B \quad (1)$$

a wrong idea that is stored in memory, or is it a reprehensible artifact that is repeatedly re-constructed by some pernicious synthesizing mechanisms? We conjecture that these faulty procedures are synthesized, not stored. The synthesis is a nearly inevitable consequence of the way that strings of procedures are concatenated. At some stage the goal

is, in some sense, to simplify

$$(\sqrt{A} + \sqrt{B})^2.$$

One possibility that will (and should) pass in review is the suggestive fact that  $(\sqrt{A})^2 = A$  and  $(\sqrt{B})^2 = B$ .

Of course, the correct identity

$$(R+S)^2 = R^2 + 2RS + S^2 \quad (2)$$

SHOULD also pass in review, and a correct evaluation procedure should note the incompatibility of (1) and (2), and should select (2) and reject (1).

But there is a further question: ought an instructional program to attempt to insert the wrong rule (1) (or some equivalent forms) deliberately into the student's memory? Experience seems to suggest that it should - with, of course, a forceful injunction NOT to use this particular rule. Unless such a frame is created, there may be no suitable site to which the negative admonition can be attached.

#### 9. When Does Creative Behavior Occur?

One would suppose, a priori, that creative behavior in mathematics would be conspicuously evident in the early stages of analyzing a newly-encountered problem. Our observations bring this supposition into question; many of the best problem solvers tackle new problems very quickly, with seeming ease, rather as if they already knew how to solve them.

But observation of student behavior after a problem has been solved (or after a problem has been interrupted and left unfinished) shows an interesting phenomenon. The best students are very often observed to be preoccupied, deep in thought, for measurably long intervals of time. Somewhat similarly, D, a 10th grade boy - one of the two best math students

in his school - has on 4 occasions after important examinations sent written notes to his teachers, correcting minor errors he made on his (excellent) exam papers. Evidently, after passing in an exam and leaving the examination room, D is able to run through his entire work on the exam - or at least major sections of it - and, if he finds errors, he can write down an accurate account of what he did on the exam paper, and write down an explanation for the error, and a correction. This is strikingly similar to the abilities of excellent chess players to replay entire games from memory after they have finished playing them - and their frequent habit of doing so.

We are finding this after-the-fact analysis a difficult phenomenon to describe in objective terms, yet we are convinced that we are observing it. Incidentally, we believe we find references to this same phenomenon in writings of Polya and of Jeremy Kilpatrick, and in a recent lecture by Allen Newell.

The contrast between different students is dramatically apparent. After working on a mathematical task the thoughtful students are often lost in apparent contemplation; other students rush to hand in exam papers or written work, visibly turning their attention to other matters even before the paper leaves their grasp.

#### 10. What Happens During Post-task Contemplation?

Taking the optimistic view that objective descriptions of the phenomenon of post-task contemplation will, sooner or later, be carried out, one can ask: what happens during this time? What information processing is being carried out?

Probably several different things occur. The example of D, the tenth-grader who sends notes to his teachers after examinations, suggests that checking over one's work for possible errors - referring to records stored, perhaps, in short term memory - is one information processing activity that probably occurs. The classical work of Kurt Lewin, Ovsiankina, and Zeigarnik suggests that further efforts to think through some of the incompleted lines of reasoning may also occur.

We conjecture that this may well be the stage where episodic memory of what has just occurred is used as the basis for analysis that adds to semantic memory. Observation of differences in student performance indicates that "thoughtful" and "hasty" students have approximately equal memory records in episodic memory - they are equally well able to tell you what happened, who suggested what, who dropped a pencil, etc. - but thoughtful students seem to exhibit great superiority in being able to discuss the problem-solution experience in abstract analytical terms, in terms of strategy, in terms of meta-language descriptors. We conjecture that this may well be the stage when VMS sequences are being consolidated into integrated sequences, when meta-language is being developed as a meaningful rhetoric for strategic planning, when meta-language descriptors are being attached to procedures and to frames, and when admonitions to avoid false rules or inadvisable procedures are being written clearly and attached to appropriate negative flags (or when the point value of different strategies is being recomputed, and related to cues that should be recognized as positive or negative indications for use or avoidance of various frames or procedures).

SUMMARY. Observations of students learning arithmetic, algebra, geometry, and calculus can be interpreted against a conceptual framework of hypothetical information-handling processes, such as "procedures," "frames," "retrieval from memory," etc. A few examples are suggested; many more are presented, in considerable detail, in Davis, Jockusch, and McKnight (1978).

While the information-handling devices are hypothetical, their use is capable of shifting the level of discussion so that questions of information processing (and obstacles to better information processing) can replace (or at least be added to) questions about specific correct or incorrect student responses. (In much the same way, hypothetical atoms, nuclei, neutrinos, etc. have shifted the level of chemistry and physics analyses from pH and precipitates to valence, hydrogen bonding, and molecular structure.)

# REFERENCES

- Bobrow, D. "Natural Language Input for a Computer Problem-Solving System," in: M. Minsky, ed., Semantic Information Processing. Cambridge, MA: MIT Press, 1968.
- Brown, John Seely and Richard R. Burton. Diagnostic Models for Procedural Bugs in Basic Mathematical Skills, in Cognitive Science, Vol. 2, No. 2. April-June, 1978.
- Davis, Robert B., Elizabeth Jockusch, and Curtis McKnight. "Cognitive Processes in Learning Algebra" in The Journal of Children's Mathematical Behavior, Vol. 2, No. 1 (1978).
- Davis, Robert B. "Representing Knowledge About Mathematics for Computer-Aided Teaching, Part I.-Educational Applications of Conceptualizations from Artificial Intelligence," in E.W. Elcock and Donald Michie, eds, Machine Intelligence 8: Machine Representations of Knowledge. Ellis Horwood, Ltd. (dist. by Halstead Press, John Wiley & Sons, Inc., New York), 1977, pp. 363-386.
- Erlwanger, Stanley H. "Benny's Conception of Rules and Answers in IPI Mathematics" The Journal of Children's Mathematical Behavior, Vol. I, No. 2 (Autumn, 1973), pp. 7-26.
- Erlwanger, Stanley H. Case Studies of Children's Conceptions of Mathematics. Doctoral Dissertation, University of Illinois, Urbana, Illinois, 1974.
- Erlwanger, Stanley H. "Case Studies of Children's Conceptions of Mathematics-Part I," The Journal of Children's Mathematical Behavior, Vol. I, No. 3 (Summer, 1975) pp. 157-283.
- Hinsley, Dan, John Hayes, and Herbert Simon. "From Words to Equations - Meaning and Representation in Algebra Word Problems," in: Marcel Just and Patricia Carpenter, eds., Cognitive Processes in Comprehension. Hillsdale, N.J.: Erlbaum, 1977.
- Kilpatrick, Jeremy. Analyzing the solution of word problems in mathematics: An exploratory study. Ph.D. thesis, Stanford University, 1967.
- Michener, Edwina R. Understanding Understanding Mathematics, Cognitive Science, Vol. 2, no. 4 (Oct.-Dec., 1978), pp. 361-383.
- Minsky, Marvin. "A Framework for Representing Knowledge," in: P. Winston, ed., The Psychology of Computer Vision. New York: McGraw-Hill, 1975.
- Minsky, Marvin, and Papert, Seymour, Artificial Intelligence Memo No. 252, Cambridge, Mass.: Massachusetts Institute of Technology Artificial Intelligence Laboratory, January 1, 1972.
- Paige, Jeffery M. and Herbert A. Simon. "Cognitive Processes in Solving Algebra Word Problems," in: B. Kleinmuntz, ed., Problem Solving: Research, Method, and Theory. New York: John Wiley, 1966. pp. 51-119.

Rumelhart, David E. and Andrew Ortony, "The Representation of Knowledge in Memory," chapter 4 in: Richard C. Anderson, Rand Spiro and William Montaque, eds., Schooling and the Acquisition of Knowledge. Hillsdale, N.J.: Lawrence Erlbaum Associates, 1977.

Schank, Roger. "Representation and Understanding of Text," chapt. 27 in: E.W. Elcock & Donald Michie, eds., Machine Intelligence 8: Machine Representations of Knowledge. Ellis Horwood Ltd. (dist. by Halstead Press, John Wiley & Sons Inc., New York), 1977, pp. 575-619.

Van Lehn, Kurt and John Seely Brown. Planning Nets: A Representation for Formalizing Analogies and Semantic Models of Procedural Skills. (to appear).

Winograd, Terry. Procedures as a Representation for Data in a Computer Program for Understanding Natural Language. Cambridge, Massachusetts: M.I.T. Artificial Intelligence Laboratory, Feb. 1971.